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Ergodic Properties of Several Interacting Poisson Particles*

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Beginning with independent Markov processes, multiparameter Markov vector processes are constructed. Stopping time vectors are used to permit stopping of the components at different times, and a strong Markov property is proved. Volkonskii's method of random change of time scale is generalized to permit simultaneous changes of time scale. These results are applied to determining the asymptotic distribution of several interacting Poisson particles in terms of the asymptotic distributions of the particles in the absence of any interaction and the speed functions producing the interaction.

1. INTRODUCTION

Systems of many particles undergoing interacting random motions in a common state space were discussed recently by Spitzer [6]. The basic idea is to start with a system of N particles which evolve independently of each other initially and use a system of speed functions, one for each particle, to modify their motions so as to produce a specified interaction. More specifically, Spitzer considers a finite state space E and an irreducible doubly stochastic kernel $P(x, y)$ on $E \times E$. In the absence of any interaction between the particles, the N particles move on trajectories of independent Markov processes with transition function $\exp t(P - I)$. If $x \neq y$, the probability that a particle at x will jump to y in a time interval of length Δt is $P(x, y)\, \Delta t + o(\Delta t)$. Letting Ψ and Φ denote strictly positive functions on E^{N-1} and E^N, respectively, and defining $\pi_n x = (x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N)$ whenever $x = (x_1, \ldots, x_N) \in E^N$, Spitzer introduces a speed function $c_n(x)$ corresponding to the n-th particle by putting

$$c_n(x) = \Psi(\pi_n x)/\Phi(x), \quad x \in E^N, \qquad 1 \leqslant n \leqslant N. \tag{1.1}$$

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The speed functions are used to describe a system of N interacting particles as follows. Suppose the position of the N particles at a given instant is $x = (x_1, \ldots, x_N)$. Then the probability that the n-th particle will jump from x_n to y_n in a time interval of length Δt is $c_n(x)\, P(x_n, y_n)\, \Delta t + o(\Delta t)$. That this description actually specifies a Markov process with just this property can be verified by defining a kernel

$$\Omega(x, y) = \begin{cases} c_n(x)\, P(x_n, \eta) & \text{if } \; y = (x_1, \ldots, x_{n-1}, \eta, x_{n+1}, \ldots, x_N) \neq x \\ & \qquad n = 1, \ldots, N, \\ -\sum_{n=1}^{N} \sum_{\eta \in E \sim \{x_n\}} c_n(x)\, P(x_n, \eta) & \text{if } \; y = x, \\ 0, & \text{otherwise,} \end{cases}$$

and constructing a Markov process $\{x_t : t \geqslant 0\}$ from the Markovian semigroup of operators $P_t = \exp t\Omega$. Assuming that the function Φ is normalized so that it can be interpreted as a probability measure on E^N, Spitzer proved that

$$\lim_{t \to \infty} P_x[x_t = y] = \Phi(y), \quad y \in E^N, \tag{1.2}$$

where P_x denotes the probability measure corresponding to the starting point x. The fact that the kernel P is doubly stochastic and irreducible plays an essential role in the proof.

It was shown in [4] that the conditions on P could be removed provided $\Psi \equiv 1$ in (1.1) and the right side of (1.2) is changed to reflect the fact that the x_t process may have more than one ergodic class and may have transient states. This was accomplished by applying Volkonskii's method of random time substitution [7]. By employing simultaneous random time substitutions, we will exhibit the asymptotic distribution of N interacting particles for arbitrary speed functions of the form (1.1) in terms of the asymptotic distributions of the noninteracting particles and the function Φ.

2. Multiparameter Markov Processes

The state space of our particles will be a locally compact Hausdorff space E having a countable base. A distinguished point Δ is adjoined to E as an isolated point if E is compact or as the point at infinity if E is not compact. We will consider $E_\Delta = E \cup \{\Delta\}$ as the state space of the particles.

$\mathscr{E}_\Delta$ will denote the σ-algebra of Borel subsets of E_Δ. The Banach space of real-valued continuous functions on E_Δ will be denoted by $C(E_\Delta)$, the norm being the usual supremum norm.

Using the notation and terminology of [1], for each positive integer $n \leqslant N$ let $\{\Omega^n, \mathscr{F}^n, \mathscr{F}_t^n, \theta_t^n, \xi_t^n, P_x^n\}$ be a normal right-continuous strong Markov process with state space E_Δ, infinite lifetime, and $\xi_\infty^n = \Delta$. We will assume that the process has been constructed so that for each $\omega \in \Omega^n$ and $u \geqslant 0$ there is an $\omega' \in \Omega^n$ such that $\xi_t^n(\omega') = \xi_{t\wedge u}^n(\omega)$ for all $t \geqslant 0$. We will also assume that each ξ_t^n process is a Feller process. Then each ξ_t^n process determines a strongly continuous Markovian semigroup $\{P_t^n: t \geqslant 0\}$ of bounded linear operators on $C(E_\Delta)$ by means of the equation

$$P_t^n f(x) = E_x^n[f(\xi_t^n)] = \int f(\xi_t^n)\, dP_x^n, \quad f \in C(E_\Delta).$$

The infinitesimal generator of the P_t^n semigroup will be denoted by A_n and its domain by $\mathscr{D}(A_n)$. In view of the compactness of E_Δ, the strong infinitesimal generator A_n and the weak infinitesimal generator of the P_t^n semigroup coincide (cf. [3, p. 143]).

Consider now a system of N independent distinguishable particles with the behavior of the n-th particle described by the process $\{\Omega^n, \mathscr{F}^n, \mathscr{F}_t^n, \theta_t^n, \xi_t^n, P_x^n\}$. The state space for the system of N particles is the product space $X = \prod_{n=1}^N E_\Delta$ endowed with the product topology. If we let $\Omega = \prod_{n=1}^N \Omega^n$, then each ξ_t^n can and will be regarded as a function on Ω so that if $\omega = \langle\omega_n\rangle = (\omega_1, \omega_2, \ldots, \omega_n) \in \Omega$, then $\xi_t^n(\omega) = \xi_t^n(\omega_n)$. The usual product σ-algebra $\prod_{n=1}^N \mathscr{F}^n$ of subsets of Ω containing all sets of the form $\prod_{n=1}^N F_n$, where $F_n \in \mathscr{F}^n$, will be denoted by $\mathscr{F}$. Each ξ_t^n is measurable relative to $\mathscr{F}$. If $x = \langle x_n\rangle \in X$, we can define a probability measure space $(\Omega, \mathscr{F}, P_x)$ as the usual product of the measure spaces $(\Omega^n, \mathscr{F}^n, P_{x_n}^n)$. Denoting expectations relative to P_x by $E_x[\cdot]$,

$$E_x\left[\prod_{n=1}^N f_n(\xi_{t_n}^n)\right] = \prod_{n=1}^N E_{x_n}^n[f_n(\xi_{t_n}^n)] \tag{2.1}$$

whenever $f_1, \ldots, f_N$ are bounded Borel measurable functions on E_Δ and $t_1, \ldots, t_N$ are nonnegative extended real numbers.

Letting R^+ denote the set of nonnegative extended real numbers, an element of $\mathbf{T} = \prod_{n=1}^N R^+$ will be denoted by $\mathbf{t} = \langle t_n\rangle$. If $\alpha \in R^+$, $\langle\alpha\rangle$ will denote the vector $(\alpha, \alpha, \ldots, \alpha)$. Algebraic and lattice operations on two elements of $\mathbf{T}$ are performed componentwise in the usual way.

If $\mathbf{s} = \langle s_n \rangle$ and $\mathbf{t} = \langle t_n \rangle$ are elements of $\mathbf{T}$, we write $\mathbf{s} < \mathbf{t}$, ($\mathbf{s} \leqslant \mathbf{t}$) if $s_n < t_n$ ($s_n \leqslant t_n$) for each n. We will now define a process indexed by elements of $\mathbf{T}$ on the probability measure space $(\Omega, \mathscr{F}, P_x)$ by allowing the parameter of the $\xi_t{}^n$ process to depend upon n; that is, we consider the N processes $\{\xi_{t_n}^n : t_n \geqslant 0\}$. For each $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$ and $\omega = \langle \omega_n \rangle \in \Omega$, we let $\xi_{\mathbf{t}}(\omega) = \langle \xi_{t_n}^n(\omega) \rangle = \langle \xi_{t_n}^n(\omega_n) \rangle$. Each $\xi_{\mathbf{t}}$ is a mapping from Ω to X and is measurable relative to $\mathscr{F}$. For each $\mathbf{t} \in \mathbf{T}$, we will let $\mathscr{F}_{\mathbf{t}} = \sigma(\xi_{s_n}^n : s_n \leqslant t_n, 1 \leqslant n \leqslant N)$, $\mathscr{F}_\infty = \sigma(\xi_{s_n}^n : s_n \in R^+, 1 \leqslant n \leqslant N)$, and $\mathscr{F}_{\mathbf{t}+} = \bigcap_{\mathbf{s} > \mathbf{t}} \mathscr{F}_{\mathbf{s}}$. Each $\xi_{\mathbf{t}}$ is clearly measurable relative to $\mathscr{F}_{\mathbf{t}}$. The shift operator $\theta_{\mathbf{t}}$, $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$, is defined for $\omega = \langle \omega_n \rangle \in \Omega$ by $\theta_{\mathbf{t}}(\omega) = \langle \theta_{t_n}^n \omega_n \rangle$. It is easily seen that $\xi_{\mathbf{t}+\mathbf{h}} = \xi_{\mathbf{t}} \circ \theta_{\mathbf{h}}$ for $\mathbf{t}, \mathbf{h} \in \mathbf{T}$.

Consider now the system $\{\Omega, \mathscr{F}, \mathscr{F}_{\mathbf{t}}, \theta_{\mathbf{t}}, \xi_{\mathbf{t}}, P_x\}$, where $\mathbf{t} \in \mathbf{T}$ and $x \in X$. This system describes the behavior of a system of N independent particles, each having its own time parameter. If for each $\mathbf{t} \in \mathbf{T}$ and each bounded Borel measurable real-valued function f on X we define

$$P_{\mathbf{t}} f(x) = E_x[f(\xi_{\mathbf{t}})] = \int f(\xi_{t_1}^1, \ldots, \xi_{t_N}^N)\, dP_x, \tag{2.2}$$

then it follows from (2.1) and the fact that each $\xi_{t_n}^n$ process is Feller that $P_{\mathbf{t}} f \in C(X)$ whenever $f = f_1 \cdot \cdots \cdot f_N$, with $f_n(x)$ a real-valued continuous function of just the n-th component x_n of $x = \langle x_n \rangle$. It follows from (2.2) and the Stone–Weierstrass theorem that $P_{\mathbf{t}} f \in C(X)$ whenever $f \in C(X)$. Clearly $P_{\mathbf{t}}$ is a bounded linear operator on $C(X)$. We will now show that the collection of operators $\{P_{\mathbf{t}} : \mathbf{t} \in \mathbf{T}\}$ is an N-parameter semigroup of operators on $C(X)$ or, more generally, that the $\xi_{\mathbf{t}}$ process is a Markov process indexed by the parameter set $\mathbf{T}$.

LEMMA 1. *For each $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$ and each bounded $\mathscr{F}_\infty$ measurable real-valued function Y on Ω, $E_x[Y \circ \theta_{\mathbf{t}} \mid \mathscr{F}_{\mathbf{t}}] = E_{\xi_{\mathbf{t}}}[Y]$ a.e. P_x.*

Proof. It suffices to prove the assertion for Y of the form $\prod_{n=1}^N Y_n$, where each Y_n is a function of just the n-th component ω_n of ω and is measurable relative to $\sigma(\xi_{s_n}^n : s_n \geqslant 0)$ (cf. [1, p. 6]); in this case

$$E_x\left[\prod_{n=1}^N Y_n \circ \theta_{\mathbf{t}} \,\middle|\, \mathscr{F}_{\mathbf{t}}\right] = E_x\left[\prod_{n=1}^N Y_n \circ \theta_{\mathbf{t}} \,\middle|\, \sigma(\xi_{s_n}^n : s_n \leqslant t_n, 1 \leqslant n \leqslant N)\right].$$

Now let $F_1, \ldots, F_N$ be Borel subsets of the reals and let $0 \leqslant s_n \leqslant t_n$,

$1 \leqslant n \leqslant N$. Using the independence of the components of the $\xi_{\mathbf{t}}$ process and (2.1)

$$\int_{\{\xi^1_{s_1}\in F_1,\ldots,\xi^N_{s_N}\in F_N\}} \prod_{n=1}^{N} Y_n \circ \theta_{\mathbf{t}}\, dP_x = \prod_{n=1}^{N} \int_{\{\xi^n_{s_n}\in F_n\}} Y_n \circ \theta_{t_n}\, dP^n_{x_n}$$

$$= \prod_{n=1}^{N} \int_{\{\xi^n_{s_n}\in F_n\}} E^n_{\xi^n_{t_n}}[Y_n]\, dP^n_{x_n}$$

$$= \int_{\{\xi^1_{s_1}\in F_1,\ldots,\xi^N_{s_N}\in F_N\}} E_{\xi_{\mathbf{t}}}\left[\prod_{n=1}^{N} Y_n\right] dP_x .$$

Since the class of sets of the form $\{\xi^1_{s_1} \in F_1, \ldots, \xi^N_{s_N} \in F_N\}$, $s_n \leqslant t_n$, generates $\mathscr{F}_{\mathbf{t}}$ and $E_{\xi_{\mathbf{t}}}[\prod_{n=1}^{N} Y_n]$ is $\mathscr{F}_{\mathbf{t}}$ measurable,

$$E_x\left[\prod_{n=1}^{N} Y_n \circ \theta_{\mathbf{t}} \,\middle|\, \mathscr{F}_{\mathbf{t}}\right] = E_{\xi_{\mathbf{t}}}\left[\prod_{n=1}^{N} Y_n\right]$$

a.e. P_x.

If $\tau_1, \ldots, \tau_N$ are extended real-valued nonnegative functions on Ω, we say that $\boldsymbol{\tau} = \langle \tau_n \rangle$ is an $\mathscr{F}_{\mathbf{t}}(\mathscr{F}_{\mathbf{t}+})$ stopping time vector if for each $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$ with real components, $\bigcap_{n=1}^{N} \{\tau_n \leqslant t_n\} \in \mathscr{F}_{\mathbf{t}}\ (\mathscr{F}_{\mathbf{t}+})$. If $\boldsymbol{\tau}$ is an $\mathscr{F}_{\mathbf{t}}\ (\mathscr{F}_{\mathbf{t}+})$ stopping time vector, we define $\mathscr{F}_{\boldsymbol{\tau}}\ (\mathscr{F}_{\boldsymbol{\tau}+})$ to be the σ-subalgebra of $\mathscr{F}_\infty$ containing all sets $A \in \mathscr{F}_\infty$ such that $A \cap \bigcap_{n=1}^{N} \{\tau_n \leqslant t_n\} \in \mathscr{F}_{\mathbf{t}}$ $(A \cap \bigcap_{n=1}^{N} \{\tau_n < t_n\} \in \mathscr{F}_{\mathbf{t}})$ for all $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$ with real components and we define $\xi_{\boldsymbol{\tau}}$ by the equation $\xi_{\boldsymbol{\tau}}(\omega) = \xi_{\boldsymbol{\tau}(\omega)}(\omega) = \langle \xi^n_{\tau_n(\omega)}(\omega) \rangle = \langle \xi^n_{\tau_n(\omega)}(\omega_n) \rangle$, $\omega = \langle \omega_n \rangle \in \Omega$. The proof of the following theorem differs only insignificantly from that in [1].

LEMMA 2. *If $\boldsymbol{\tau}$ is an $\mathscr{F}_{\mathbf{t}}\ (\mathscr{F}_{\mathbf{t}+})$ stopping time vector, then $\boldsymbol{\tau}$ and $\xi_{\boldsymbol{\tau}}$ are $\mathscr{F}_{\boldsymbol{\tau}}(\mathscr{F}_{\boldsymbol{\tau}+})$ measurable.*

Proof. We will prove the assertion concerning $\mathscr{F}_{\mathbf{t}+}$ stopping time vectors, the other case being similar. Let $\boldsymbol{\tau} = \langle \tau_n \rangle$ be an $\mathscr{F}_{\mathbf{t}+}$ stopping time vector. This is the case if and only if $\bigcap_{n=1}^{N} \{\tau_n < t_n\} \in \mathscr{F}_{\mathbf{t}}$ for all $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$ with real components (cf. [1, p. 31]). To show that $\boldsymbol{\tau}$ is $\mathscr{F}_{\boldsymbol{\tau}+}$ measurable, it suffices to show that each component τ_j of $\boldsymbol{\tau}$ is $\mathscr{F}_{\boldsymbol{\tau}+}$ measurable. Let s_j be a nonnegative real number and let $\mathbf{t} = \langle t_n \rangle$ be an element in $\mathbf{T}$ with real components. If $s_j \leqslant t_j$, then

$$\{\tau_j < s_j\} \cap \bigcap_{n=1}^{N} \{\tau_n < t_n\} = \bigcap_{n=1}^{N} \{\tau_n < t_n'\},$$

where $\mathbf{t}' = \langle t_n' \rangle$ is obtained from $\mathbf{t} = \langle t_n \rangle$ by replacing t_j by s_j. Since $\bigcap_{n=1}^N \{\tau_n < t_n'\} \in \mathscr{F}_{\mathbf{t}'} \subset \mathscr{F}_{\mathbf{t}}$, $\{\tau_j < s_j\} \cap \bigcap_{n=1}^N \{\tau_n < t_n\} \in \mathscr{F}_{\mathbf{t}}$. If $s_j > t_j$, then $\{\tau_j < s_j\} \cap \bigcap_{n=1}^N \{\tau_n < t_n\} = \bigcap_{n=1}^N \{\tau_n < t_n\} \in \mathscr{F}_{\mathbf{t}}$. Since $\mathbf{t}$ is an arbitrary element of $\mathbf{T}$ with real components, $\{\tau_j < s_j\} \in \mathscr{F}_{\boldsymbol{\tau}+}$ for each real number s_j and τ_j is $\mathscr{F}_{\boldsymbol{\tau}+}$ measurable. We now show that $\xi_{\boldsymbol{\tau}}$ is $\mathscr{F}_{\boldsymbol{\tau}+}$ measurable essentially using the proof in [1]. Note first of all that $\xi_{\mathbf{t}}(\omega) = \langle \xi_{t_n}^n(\omega) \rangle$ is jointly right-continuous in $t_1, ..., t_N$ for each $\omega \in \Omega$. This implies that for each $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$ with real components, the map $\Phi_{\mathbf{t}}$ from $\prod_{n=1}^N [0, t_n) \times \Omega$ to X defined by $\Phi_{\mathbf{t}}(\mathbf{s}, \omega) = \xi_{\mathbf{s}}(\omega)$ is measurable relative to $\mathscr{R}_{\mathbf{t}} \times \mathscr{F}_{\mathbf{t}}$ where $\mathscr{R}_{\mathbf{t}}$ is the collection of Borel subsets of $\prod_{n=1}^N [0, t_n)$ Now let $\Psi_{\mathbf{t}}$ be the map from $\bigcap_{n=1}^N \{\tau_n < t_n\}$ to $\prod_{n=1}^N [0, t_n) \times \Omega$ defined by $\Psi_{\mathbf{t}}(\omega) = (\boldsymbol{\tau}(\omega), \omega)$. Now $\Psi_{\mathbf{t}}^{-1}$ maps elements of $\mathscr{R}_{\mathbf{t}} \times \mathscr{F}_{\mathbf{t}}$ into elements of $\mathscr{F}_{\mathbf{t}}$ for if $\mathbf{s} = \langle s_n \rangle \leqslant \mathbf{t}$ and $\Lambda \in \mathscr{F}_{\mathbf{t}}$, then $\Psi_{\mathbf{t}}^{-1}(\prod_{n=1}^N [0, s_n) \times \Lambda) = \bigcap_{n=1}^N \{\tau_n < s_n\} \cap \Lambda \in \mathscr{F}_{\mathbf{t}}$. Suppose Ψ is the restriction of $\xi_{\boldsymbol{\tau}}$ to $\bigcap_{n=1}^N \{\tau_n < t_n\}$. Then $\Psi = \Phi_{\mathbf{t}} \circ \Psi_{\mathbf{t}}$ and $\Psi^{-1}(B) \in \mathscr{F}_{\mathbf{t}}$ for any Borel set $B \subset X$. Therefore, $\{\xi_{\boldsymbol{\tau}} \in B\} \cap \bigcap_{n=1}^N \{\tau_n < t_n\} = \Psi^{-1}(B) \in \mathscr{F}_{\mathbf{t}}$ for any Borel set $B \subset X$, and $\xi_{\boldsymbol{\tau}}$ is $\mathscr{F}_{\boldsymbol{\tau}+}$ measurable.

We will henceforth adopt the convention that any real-valued function on E_Δ vanishes at Δ. In particular, if f is a real-valued function on X and some component of $x = \langle x_n \rangle$ is equal to Δ, then $f(x) = 0$.

LEMMA 3. *If f is a bounded Borel measurable function on X, $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$ and $\boldsymbol{\tau} = \langle \tau_n \rangle$ is an $\mathscr{F}_{\mathbf{t}+}$ stopping time vector, then* $E_x[f(\xi_{\boldsymbol{\tau}+\mathbf{t}}) \mid \mathscr{F}_{\boldsymbol{\tau}+}] = E_{\xi_{\boldsymbol{\tau}}}[f(\xi_{\mathbf{t}})]$ a.e. P_x.

Proof. It suffices to prove the result for functions f of the form $f = \prod_{n=1}^N f_n$, where each f_n is a continuous function of just the n-th component x_n of x. If $\boldsymbol{\alpha} = \langle \alpha_n \rangle$ is an element of $\mathbf{T}$ with real components, we will let $\boldsymbol{\alpha} \cdot \mathbf{t} = \sum_{n=1}^N \alpha_n t_n$. Let A be an arbitrary element of $\mathscr{F}_{\boldsymbol{\tau}+}$. Then

$$E_x \left[\int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} f(\xi_{\boldsymbol{\tau}+\mathbf{t}})\, dt_1, ..., dt_N \,;\, A \right]$$

$$= E_x \left[\int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} \prod_{n=1}^N f_n(\xi_{\tau_n+t_n}^n)\, dt_1, ..., dt_N \,;\, A \right].$$

For each positive k let

$$\varphi_k(t) = \begin{cases} (i+1)/2^k & \text{if} \quad i/2^k \leqslant t < (i+1)/2^k, \qquad i \geqslant 0 \\ +\infty & \text{if} \quad t = +\infty \end{cases}$$

and let $\varphi_k(\boldsymbol{\tau}) = \langle \varphi_k(\tau_n) \rangle$. Then $\varphi_k(\boldsymbol{\tau})$ is an $\mathscr{F}_{\mathbf{t}+}$ stopping time vector, $\tau_n \leqslant \varphi_k(\tau_n)$, and $\varphi_k(\tau_n)$ decreases to τ_n as k increases. Therefore,

$$E_x \left[\int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} f(\xi_{\boldsymbol{\tau}+\mathbf{t}})\, dt_1 ,\dots, dt_N \,;\, A \right]$$

$$= \lim_{k\to\infty} E_x \left[\int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} \prod_{n=1}^N f_n(\xi^n_{\varphi_k(\tau_n)+t_n})\, dt_1 ,\dots, dt_N \,;\, A \right]$$

$$= \lim_{k\to\infty} \sum_{i_1,\dots,i_N} E_x \left[\int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} \prod_{n=1}^N f_n(\xi^n_{\varphi_k(\tau_n)+t_n})\, dt_1 ,\dots, dt_N \,; \right.$$

$$\left. A \cap \bigcap_{n=1}^N \{\varphi_k(\tau_n) = i_n/2^k\} \right]$$

$$= \lim_{k\to\infty} \sum_{i_1,\dots,i_N} \int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} E_x \left[\prod_{n=1}^N f_n(\xi^n_{\varphi_k(\tau_n)+t_n}); \right.$$

$$\left. A \cap \bigcap_{n=1}^N \{\varphi_k(\tau_n) = i_n/2^k\} \right] dt_1 ,\dots, dt_N \,.$$

Fixing $i_1 ,\dots, i_N$ and letting $\mathbf{t}_k = \langle i_n/2^k \rangle$,

$$A \cap \bigcap_{n=1}^N \{\varphi_k(\tau_n) = i_n/2^k\} = A \cap \bigcap_{n=1}^N \{(i_n - 1)/2^k \leqslant \tau_n < i_n/2^k\} \in \mathscr{F}_{\mathbf{t}_k} .$$

It follows from Lemma 1 and (2.1) that

$$E_x \left[\int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} f(\xi_{\boldsymbol{\tau}+\mathbf{t}})\, dt_1 ,\dots, dt_N \,;\, A \right]$$

$$= \lim_{k\to\infty} \sum_{i_1,\dots,i_N} \int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} E_x \left[E_{\xi_{\mathbf{t}_k}} \left[\prod_{n=1}^N f_n(\xi^n_{t_n}) \right]; \right.$$

$$\left. A \cap \bigcap_{n=1}^N \{\varphi_k(\tau_n) = i_n/2^k\} \right] dt_1 ,\dots, dt_N$$

$$= \lim_{k\to\infty} \int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} E_x \left[E_{\xi_{\varphi_k(\boldsymbol{\tau})}} \left[\prod_{n=1}^N f_n(\xi^n_{t_n}) \right]; A \right] dt_1 ,\dots, dt_N$$

$$= \lim_{k\to\infty} \int_0^\infty \cdots \int_0^\infty e^{-\boldsymbol{\alpha}\cdot\mathbf{t}} E_x \left[\prod_{n=1}^N E^{x_n}_{\xi_{\varphi_k(\tau_n)}} [f_n(\xi^n_{t_n})]; A \right] dt_1 ,\dots, dt_N \,.$$

Since $E_{x_n}^n[f_n(\xi_{t_n}^n)]$ is a continuous function of x_n and $\lim_{k\to\infty} \xi_{\varphi_k(\tau_n)}^n = \xi_{\tau_n}^n$, the limit can be taken under the integral sign to obtain

$$\int_0^\infty \cdots \int_0^\infty e^{-\alpha\cdot\mathbf{t}} E_x[f(\xi_{\tau+\mathbf{t}}); A]\, dt_1, \ldots, dt_N$$

$$= \int_0^\infty \cdots \int_0^\infty e^{-\alpha\cdot\mathbf{t}} E_x[E_{\xi_\tau}[f(\xi_{\mathbf{t}})]; A]\, dt_1, \ldots, dt_N .$$

It follows that $E_x[f(\xi_{\tau+\mathbf{t}}); A] = E_x[E_{\xi_\tau}[f(\xi_{\mathbf{t}})]; A]$ for all $A \in \mathscr{F}_{\tau+}$ by the essential uniqueness of the Laplace transform and right-continuity of the components ξ_t^n.

If τ is a stopping time vector, the random shift operator θ_τ is defined for $\omega = \langle\omega_n\rangle \in \Omega$ by $\theta_\tau(\omega) = \langle\theta_{\tau_n(\omega)}^n \omega_n\rangle$. In as much as the proof of the following theorem is precisely the same as that of the corresponding theorem in [1], its proof will be omitted.

THEOREM 4. *If τ is an $\mathscr{F}_{\mathbf{t}+}$ stopping time vector and Y is a bounded $\mathscr{F}_\infty$ measurable real-valued function on Ω, then $E_x[Y \circ \theta_\tau \mid \mathscr{F}_{\tau+}] = E_{\xi_\tau}[Y]$* a.e. P_x.

Recall that the infinitesimal generator of the semigroup associated with the ξ_t^n process is A_n with domain $\mathscr{D}(A_n) \subset C(E_\Delta)$. The operator A_n has a natural interpretation as an operator on $C(X)$ as follows. Suppose $f \in C(X)$ has the property that $f(x_1, \ldots, x_{n-1}, \cdot, x_{n+1}, \ldots, x_N)$ is in the domain of A_n with $x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N$ fixed. Then $A_n f(x_1, \ldots, x_{n-1}, \cdot, x_{n+1}, \ldots, x_N)(x_n)$ is defined and will be denoted by $A_n f(x_1, \ldots, x_n, \ldots, x_N)$ or simply $A_n f$; as a function of x_n, with $x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N$ fixed, $A_n f$ is a continuous function of x_n. Consider now a function $f_n \in C(X)$ which is a function of just the n-th component x_n of x. Then f_n can and will be regarded as a function on E_Δ. If $f_n \in D(A_n)$ in addition, then the function $h_{n,\lambda} = \lambda f_n - A_n f_n$, $\lambda > 0$, has the property that

$$f_n(x_n) = \int_0^\infty e^{-\lambda t} P_t^n h_{n,\lambda}(x_n)\, dt = E_{x_n}^n \int_0^\infty e^{-\lambda t} h_{n,\lambda}(\xi_t^n)\, dt. \tag{2.3}$$

This well-known result from the theory of semigroups of operators is needed for the proof of the next theorem.

THEOREM 5. *If $f = \prod_{n=1}^N f_n$ with each f_n a function of just the n-th* *component* x_n *of* x, $f_n \in \mathscr{D}(A_n)$, *and* $\boldsymbol{\tau} = \langle \tau_n \rangle$ *is an* $\mathscr{F}_{t+}$ *stopping time vector with* $E_x[\tau_n] < +\infty$ *for each* n, *then*

$$f(x) = \sum_{n=1}^{N} E_x \left\{ \prod_{\substack{m=1 \\ m \neq n}}^{N} f_m(\xi_\tau) \left(\int_0^{\tau_n} - A_n f_n(\xi_t{}^n)\, dt \right) \right\}$$

$$+ \sum_{\substack{m_1, m_2 = 1 \\ m_1 \neq m_2}}^{N} E_x \left\{ \prod_{\substack{m=1 \\ m \neq m_1 \\ m \neq m_2}}^{N} f_m(\xi_\tau) \left(\int_0^{\tau_{m_1}} - A_{m_1} f_{m_1}(\xi_{t_1}^{m_1})\, dt_1 \right) \left(\int_0^{\tau_{m_2}} - A_{m_2} f_{m_2}(\xi_{t_2}^{m_2})\, dt_2 \right) \right\}$$

$$\vdots$$

$$+ E_x \left[\prod_{n=1}^{N} f_n(\xi_\tau) \right].$$

Proof. Consider the functions $h_{n,\lambda} = \lambda f_n - A_n f_n$, $\lambda > 0$. By (2.3), independence of the components of the ξ_t process, and (2.1),

$$\begin{aligned} f(x) &= \prod_{n=1}^{N} E_{x_n}^n \int_0^\infty e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n \\ &= E_x \left[\prod_{n=1}^{N} \int_0^\infty e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n \right] \\ &= \sum_{I_1, \ldots, I_N} E_x \left[\prod_{n=1}^{N} \int_{I_n} e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n \right], \end{aligned}$$

where I_n is either $[0, \tau_n)$ or $[\tau_n, \infty)$, $n = 1, \ldots, N$. In order to avoid cumbersome notation, we will only consider particular terms in the latter sum. The other terms are handled in the same way.

Case A. Suppose $I_1 = [0, \tau_1)$ and $I_j = [\tau_j, \infty)$, $j = 2, \ldots, N$. Then

$$\begin{aligned} &E_x \left[\prod_{n=1}^{N} \int_{I_n} e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n \right] \\ &\quad = E_x \left[E_x \left[\left(\int_0^{\tau_1} e^{-\lambda t_1} h_{1,\lambda}(\xi_{t_1}^1)\, dt_1 \right) \prod_{n=2}^{N} \int_{\tau_n}^\infty e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n \mid \mathscr{F}_{\tau+} \right] \right]. \end{aligned}$$

Since $\int_0^{\tau_1} e^{-\lambda t_1} h_{1,\lambda}(\xi_{t_1}^1)\, dt_1$ is measurable relative to $\mathscr{F}_{\tau+}$,

$$E_x\left[\prod_{n=1}^N \int_{I_n} e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n\right]$$

$$= E_x\left[\left(\int_0^{\tau_1} e^{-\lambda t_1} h_{1,\lambda}(\xi_{t_1}^1)\, dt_1\right) E_x\left[\prod_{n=2}^N e^{-\lambda\tau_n} \int_0^\infty e^{-\lambda s_n} h_{n,\lambda}(\xi_{s_n}^n) \circ \theta_\tau\, ds_n \mid \mathscr{F}_{\tau+}\right]\right]$$

$$= E_x\left[\left(\int_0^{\tau_1} e^{-\lambda t_1} h_{1,\lambda}(\xi_{t_1}^1)\, dt_1\right) e^{-\lambda\tau_1 - \cdots - \lambda\tau_N} \prod_{n=2}^N f_n(\xi_\tau)\right]$$

by Theorem 4. Using the fact that $E_x[\tau_n] < \infty$, $n = 1,\ldots, N$,

$$\lim_{\lambda\to 0} E_x\left[\prod_{n=1}^N \int_{I_n} e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n\right] = E_x\left[\prod_{n=2}^N f_n(\xi_\tau)\left(\int_0^{\tau_1} - A_1 f_1(\xi_{t_1}^1)\, dt_1\right)\right].$$

Case B. Suppose $I_1 = [0, \tau_1)$, $I_2 = [0, \tau_2)$, and $I_n = [\tau_n, \infty)$, $n = 3,\ldots, N$. In this case, $\int_0^{\tau_j} e^{-\lambda t_j} h_{j,\lambda}(\xi_{t_j}^j)\, dt_j$, $j = 1, 2$, are both measurable relative to $\mathscr{F}_{\tau+}$ and as before

$$E_x\left[\prod_{n=1}^N \int_{I_n} e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n\right]$$

$$= E_x\left[\left(\int_0^{\tau_1} e^{-\lambda t_1} h_{1,\lambda}(\xi_{t_1}^1)\, dt_1\right)\left(\int_0^{\tau_2} e^{-\lambda t_2} h_{2,\lambda}(\xi_{t_2}^2)\, dt_2\right) \cdot e^{-\lambda\tau_3 - \cdots - \lambda\tau_N} \prod_{n=3}^N f_n(\xi_\tau)\right].$$

Therefore,

$$\lim_{\lambda\to 0} E_x\left[\prod_{n=1}^N \int_{I_n} e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n\right]$$

$$= E_x\left[\left(\int_0^{\tau_1} - A_1 f_1(\xi_{t_1}^1)\, dt_1\right)\left(\int_0^{\tau_2} - A_2 f_2(\xi_{t_2}^2)\, dt_2\right) \prod_{n=3}^N f_n(\xi_\tau)\right].$$

Case C. Suppose $I_n = [\tau_n, \infty)$, $n = 1,\ldots, N$. In this case,

$$E_x\left[\prod_{n=1}^N \int_{I_n} e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n\right] = E_x\left[e^{-\lambda\tau_1 - \cdots - \lambda\tau_N} \prod_{n=1}^N f_n(\xi_\tau)\right]$$

and

$$\lim_{\lambda\to 0} E_x\left[\prod_{n=1}^N \int_{I_n} e^{-\lambda t_n} h_{n,\lambda}(\xi_{t_n}^n)\, dt_n\right] = E_x\left[\prod_{n=1}^N f_n(\xi_\tau)\right].$$

3. Simultaneous Random Time Substitutions

We will consider now the effect of simultaneously changing the time scales of the components of the ξ_t process. This was first done for a single component process by Volkonskii [7].

Throughout this section we will consider a family $\boldsymbol{\tau}(t) = \langle \tau_n(t) \rangle$, $t \geqslant 0$, of $\mathscr{F}_{\mathbf{t}+}$ stopping time vectors having the following properties for $n = 1,..., N$:

$$\begin{aligned} &\text{(i)} \quad \tau_n(s) \leqslant \tau_n(t) < +\infty \qquad \text{for all} \quad 0 \leqslant s \leqslant t < +\infty, \\ &\text{(ii)} \quad \tau_n(t+h) = \tau_n(t) + \tau_n(h) \circ \theta_{\boldsymbol{\tau}(t)} \qquad \text{for all} \quad t, h \geqslant 0, \qquad (3.1) \\ &\text{(iii)} \quad \tau_n(t) \text{ is right continuous.} \end{aligned}$$

We will always assume that $\tau_n(\infty) = +\infty$, $n = 1,..., N$. We will also define an η_t process by putting $\eta_t = \langle \eta_t^n \rangle = \xi_{\boldsymbol{\tau}(t)} = \langle \xi^n_{\tau_n(t)} \rangle$. If φ is an $\mathscr{F}_{\boldsymbol{\tau}(t)+}$ stopping time, we define $\boldsymbol{\tau}(\varphi)$ by the equation

$$\boldsymbol{\tau}(\varphi)(\omega) = \boldsymbol{\tau}(\varphi(\omega))(\omega) = \langle \tau_n(\varphi(\omega))(\omega) \rangle.$$

By (3.1) (ii), $\tau_n(\varphi + h) = \tau_n(\varphi) + \tau_n(h) \circ \theta_{\boldsymbol{\tau}(\varphi)}$. The proof of the following lemma is a modification of one found in [7].

Lemma 6. *If φ is an $\mathscr{F}_{\boldsymbol{\tau}(t)+}$ stopping time, then $\boldsymbol{\tau}(\varphi)$ is an $\mathscr{F}_{\mathbf{t}+}$ stopping time vector.*

Proof. We must show that $\bigcap_{n=1}^N \{\tau_n(\varphi) < t_n\} \in \mathscr{F}_{\mathbf{t}}$ for each $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$. Letting Q denote the set of nonnegative rationals, by the right-continuity of each $\tau_n(t)$,

$$\begin{aligned} \bigcap_{n=1}^N \{\tau_n(\varphi) < t_n\} &= \bigcap_{n=1}^N \{\varphi < \sup_{\tau_n(s) < t_n} s\} \\ &= \{\varphi < \min_{1 \leqslant n \leqslant N} (\sup_{\tau_n(s) < t_n} s)\} \\ &= \bigcup_{r \in Q} [\{\varphi < r\} \cap \{r < \min_{1 \leqslant n \leqslant N} (\sup_{\tau_n(s) < t_n} s)\}] \\ &= \bigcup_{r \in Q} [\{\varphi < r\} \cap \bigcap_{n=1}^N \{\tau_n(r) < t_n\}]. \end{aligned}$$

Since φ is an $F_{\tau(t)+}$ stopping time,

$$\{\varphi < r\} \in \mathscr{F}_{\tau(r)} \qquad \text{and} \qquad \{\varphi < r\} \cap \bigcap_{n=1}^{N} \{\tau_n(r) < t_n\} \in \mathscr{F}_{\mathbf{t}}.$$

Therefore, $\bigcap_{n=1}^{N} \{\tau_n(\varphi) < t_n\} \in \mathscr{F}_{\mathbf{t}}$ for all $\mathbf{t} = \langle t_n \rangle \in \mathbf{T}$.

LEMMA 7. *The system* $\{\Omega, \mathscr{F}, \mathscr{F}_{\tau(t)+}, \eta_t, P_x\}$ *is a strong Markov process with right continuous paths.*

Proof. Let φ be a $\mathscr{F}_{\tau(t)+}$ stopping time. Then $\boldsymbol{\tau}(\varphi)$ is an $\mathscr{F}_{\mathbf{t}+}$ stopping time vector. Let f be a bounded Borel measurable function on X. By (ii) of (3.1) and Theorem 4,

$$\begin{aligned} E_x[f(\eta_{\varphi+s}) \mid \mathscr{F}_{\tau(\varphi)+}] &= E_x[f(\xi^1_{\tau_1(\varphi+s)}, \ldots, \xi^N_{\tau_N(\varphi+s)}) \mid \mathscr{F}_{\tau(\varphi)+}] \\ &= E_x[f(\xi^1_{\tau_1(\varphi)+\tau_1(s)\circ\theta_{\tau(\varphi)}}, \ldots, \xi^N_{\tau_N(\varphi)+\tau_N(s)\circ\theta_{\tau(\varphi)}}) \mid \mathscr{F}_{\tau(\varphi)+}] \\ &= E_x[f(\xi^1_{\tau_1(s)}, \ldots, \xi^N_{\tau_N(s)}) \circ \theta_{\tau(\varphi)} \mid \mathscr{F}_{\tau(\varphi)+}] \\ &= E_{\xi_{\tau(\varphi)}}[f(\eta_s)] \\ &= E_{\eta_\varphi}[f(\eta_s)] \end{aligned}$$

a.e. P_x.

4. SPEED FUNCTIONS

We will consider now a system of simultaneous random time substitutions induced by a system of speed functions. This will require that we impose the conditions that the processes $\{\xi_t^n : t \geq 0\}$, $n = 1, \ldots, N$ are step processes and that the associated infinitesimal generators A_n are bounded operators on $C(E_\Delta)$. The first condition means that the sample functions of the processes are step functions with the set of points of discontinuity having no accumulation points.

Starting with a process $\{x_t : t \geq 0\}$ with associated infinitesimal generator A and a strictly positive function Φ, Volkonskii [7] constructed a process $\{y_t : t \geq 0\}$ with associated infinitesimal generator ΦA by means of a random time substitution $\tau(t)$ defined by the equation

$$\int_0^{\tau(t)} \frac{1}{\Phi(x_s)}\, ds = t \tag{4.1}$$

and putting $y_t = x_{\tau(t)}$. This method of defining the random time substitution $\tau(t)$ can not be used when dealing with a system of processes. However, if (4.1) is recast as the initial value problem

$$\begin{cases} \dot{\tau}(t) = \Phi(x_{\tau(t)}) & t \geqslant 0 \\ \tau(0) = 0, \end{cases}$$

then there is a natural extension of the initial value problem applicable to systems of time substitutions.

Let $c_1, \ldots, c_N$ be strictly positive continuous functions on X, let m, M be positive constants such that

$$0 < m \leqslant c_n \leqslant M \qquad n = 1, \ldots, N.$$

and let $\mathbf{a} = \langle a_n \rangle \in \mathbf{T}$. Consider the initial value problem

$$\begin{aligned} \dot{\tau}_n(t) &= c_n(\xi^1_{\tau_1(t)}, \ldots, \xi^N_{\tau_N(t)}) \qquad t \geqslant 0, \\ \tau_n(0) &= a_n, \qquad n = 1, \ldots, N. \end{aligned} \tag{4.2}$$

The first equation of (4.2) is only required to hold for all t except possibly for a finite set of points, possibly depending upon ω, in each finite interval. We will show that this system of equations has a unique solution for each $\omega \in \Omega$, that each $\boldsymbol{\tau}(t) = \langle \tau_n(t) \rangle$ is a stopping time vector, and that the $\tau_n(t)$ satisfy the conditions of (3.1).

For each positive integer $j \geqslant 1$, we define functions $\alpha_{n,j}$ on $[0, \infty)$ as follows. Consider the dyadic rationals $k2^{-j}$, $k = 0, 1, 2, \ldots$. We put

$$\alpha_{n,j}(0) = a_n,$$

and for $0 < t \leqslant 2^{-j}$

$$\alpha_{n,j}(t) = a_n + c_n(\xi^1_{a_1}, \ldots, \xi^N_{a_N}).$$

Having defined $\alpha_{1,j}(2^{-j}), \ldots, \alpha_{N,j}(2^{-j})$ in this way, we define

$$\alpha_{n,j}(t) = \alpha_{n,j}(2^{-j}) + c_n(\xi^1_{\alpha_{1,j}(2^{-j})}, \ldots, \xi^N_{\alpha_{N,j}(2^{-j})})(t - 2^{-j})$$

for $2^{-j} < t \leqslant 2 \cdot 2^{-j}$. More generally, assuming that the $\alpha_{n,j}$ are defined for $0 \leqslant t \leqslant k \cdot 2^{-j}$, we now define $\alpha_{n,j}$ on $k2^{-j} < t \leqslant (k+1)\,2^{-j}$ by putting

$$\alpha_{n,j}(t) = \alpha_{n,j}(k2^{-j}) + c_n(\xi^1_{\alpha_{1,j}(k2^{-j})}, \ldots, \xi^N_{\alpha_{N,j}(k2^{-j})})(t - k2^{-j}).$$

If we put

$$\beta_{n,j}(t) = c_n(\xi^1_{\alpha_{1,j}(k2^{-j})}, \ldots, \xi^N_{\alpha_{N,j}(k2^{-j})})$$

for $k2^{-j} < t \leqslant (k+1)\,2^{-j}$, $k = 0, 1, 2, \ldots$, then

$$\alpha_{n,j}(t) = a_n + \int_0^t \beta_{n,j}(s)\,ds. \tag{4.3}$$

Each $\alpha_{n,j}$ is continuous and strictly increasing on $[0, \infty)$, $\alpha_{n,j}(0) = a_n$, and

$$m \leqslant \frac{\alpha_{n,j}(t) - \alpha_{n,j}(s)}{t - s} \leqslant M \tag{4.4}$$

whenever $0 \leqslant s < t < \infty$. We now choose a subsequence j_i of the integers such that $\lim_{i\to\infty} \alpha_{n,j_i}(t)$ is defined for all rational t. Using (4.4), it is easy to show that

$$\tau_n(t) = \lim_{i\to\infty} \alpha_{n,j_i}(t)$$

is defined for all t and is strictly increasing on $[0, \infty)$. Moreover,

$$\lim_{i\to\infty} \beta_{n,j_i}(t) = c_n(\xi^1_{\tau_1(t)}, \ldots, \xi^N_{\tau_N(t)})$$

except possibly for a finite number of t in any compact interval, possibly depending upon $\omega \in \Omega$. It follows from (4.3) that

$$\tau_n(t) = a_n + \int_0^t c_n(\xi^1_{\tau_1(s)}, \ldots, \xi^N_{\tau_N(s)})\,ds$$

for each $t \geqslant 0$ and $n = 1, \ldots, N$. The system $\boldsymbol{\tau}(t) = \langle \tau_n(t) \rangle$ therefore solves the initial value problem (4.2). We now prove the uniqueness of the solution.

Suppose $\boldsymbol{\sigma}(t) = \langle \sigma_n(t) \rangle$ is a second such solution of (4.2). It suffices to show that $\boldsymbol{\tau}(t) = \boldsymbol{\sigma}(t)$ on each finite interval $[0, a]$. Choose b such that $\tau_n(t) \leqslant b$ and $\sigma_n(t) \leqslant b$ for $t \in [0, a]$. Now each of the N paths $\xi_t^n(\omega)$ can have at most finitely many points of discontinuity in $[0, b]$. Suppose the points of discontinuity of $\xi_t^n(\omega)$ in $[0, b]$ greater than a_n, $n = 1, \ldots, N$, are pooled and labeled so that $0 < s_1 < s_2 < \cdots < s_p < b$. Now let u_1 and v_1 be the largest values of t such that $\tau_n(t) \leqslant s_1$ and $\sigma_n(t) \leqslant s_1$ for all n, respectively. Since $\xi_\cdot^n(\omega)$ is a step function, for $t < u_1 \wedge v_1$, $\xi^n_{\tau_n(t)} = \xi^n_{a_n} = \xi^n_{\sigma_n(t)}$ and therefore $\dot{\tau}_n(t) = c_n(\xi^1_{a_1}, \ldots, \xi^N_{a_N}) =$

$\dot{\sigma}_n(t)$ for $t < u_1 \wedge v_1$. It follows that $\tau_n(t) = \sigma_n(t) = a_n + c_n(\xi_{a_1}^1, \ldots, \xi_{a_N}^N)t$ for $t \leqslant u_1 \wedge v_1$. It is easily seen that $u_1 = v_1$. Now let u_2 and v_2 be the largest values of t such that $\tau_n(t) \leqslant s_2$ and $\sigma_n(t) \leqslant s_2$ for all n, respectively. Then $u_1 < u_2 \wedge v_2$ and for $u_1 \leqslant t < u_2 \wedge v_2$, $\xi^n_{\tau_n(t)} = \xi^n_{\tau_n(u_1)} = \xi^n_{\sigma_n(t)}$. Therefore,

$$\dot{\tau}_n(t) = c_n(\xi^1_{\tau_1(u_1)}, \ldots, \xi^N_{\tau_N(u_n)}) = \dot{\sigma}_n(t)$$

for $u_1 \leqslant t < u_2 \wedge v_2$. Therefore, $\tau_n(t) = \sigma_n(t)$ for $u_1 \leqslant t < u_2 \wedge v_2$ since they agree at u_1. As before, $u_2 = v_2$. Proceeding in this way until all s_j are exhausted, we get $\tau_n(t) = \sigma_n(t)$ for all $t \in [0, a]$; that is, the solution of (4.2) is unique.

Unless otherwise specified, it will be assumed that the a_n of the initial value problem (4.2) are all zero.

LEMMA 8. *If* $t \geqslant 0$, $\mathbf{u} = \langle u_n \rangle \in \mathbf{T}$, $\omega \in \bigcap_{n=1}^N \{\tau_n(t) \leqslant u_n\}$, $\tilde{\omega} \in \Omega$, *and* $\xi_{\mathbf{s}}(\omega) = \xi_{\mathbf{s}}(\tilde{\omega})$ *for all* $\mathbf{s} \leqslant \mathbf{u}$, *then* $\tilde{\omega} \in \bigcap_{n=1}^N \{\tau_n(t) \leqslant u_n\}$.

Proof. Since $\tau_n(0) = 0$ for all n and $\bigcap_{n=1}^N \{\tau_n(0) \leqslant u_n\} = \Omega$, we can assume that $t > 0$. Let s_1 and $\tilde{s}_1$ be the times of the first discontinuity of $\xi_{\tau(s)(\omega)}(\omega)$ and $\xi_{\tau(s)(\tilde{\omega})}(\tilde{\omega})$ on $[0, t]$, respectively. Then

$$\tau_n(s)(\omega) = c_n(\xi_0{}^1(\omega), \ldots, \xi_0{}^N(\omega))s, \qquad s \leqslant s_1$$
$$\tau_n(s)(\tilde{\omega}) = c_n(\xi_0{}^1(\tilde{\omega}), \ldots, \xi_0{}^N(\tilde{\omega}))s, \qquad s \leqslant \tilde{s}_1 .$$

Since $\xi_0(\omega) = \xi_0(\tilde{\omega})$, $\tau_n(s)(\omega) = \tau_n(s)(\tilde{\omega})$ for $s \leqslant s_1 \wedge \tilde{s}_1$. Suppose $s_1 < \tilde{s}_1$. We would then have

$$\lim_{s \to s_1^-} \xi_{\tau(s)(\omega)}(\omega) \neq \xi_{\tau(s_1)(\omega)}(\omega),$$
$$\lim_{s \to s_1^-} \xi_{\tau(s)(\tilde{\omega})}(\tilde{\omega}) = \xi_{\tau(s_1)(\tilde{\omega})}(\tilde{\omega}).$$

But since $\tau(s_1)(\omega) = \tau(s_1)(\tilde{\omega}) \leqslant \mathbf{u}$, $\xi_{\tau(s)(\omega)}(\omega) = \xi_{\tau(s)(\tilde{\omega})}(\tilde{\omega})$ for $s \leqslant s_1$, a contradiction. This shows that $\tilde{s}_1 \leqslant s_1$. Similarly, $s_1 \leqslant \tilde{s}_1$, and the two are equal. By letting s_2 and $\tilde{s}_2$ be the times of the second discontinuities of $\xi_{\tau(s)(\omega)}(\omega)$ and $\xi_{\tau(s)(\tilde{\omega})}(\tilde{\omega})$ on $[0, t]$, respectively, and repeating the above argument, etc., we can show that $\tau(s)(\omega)$ and $\tau(s)(\tilde{\omega})$ agree on $[0, t]$. In particular, $\tilde{\omega} \in \bigcap_{n=1}^N \{\tau_n(t) \leqslant u_n\}$.

The first part of the proof of the following lemma is standard (cf. [2]) and is included for the sake of completeness.

THEOREM 9. *If* $\boldsymbol{\tau}(t)$ *is the unique solution of* (4.2) *satisfying* $\boldsymbol{\tau}(0) = \mathbf{0}$, *then each* $\boldsymbol{\tau}(t)$ *is an* $\mathscr{F}_{\mathbf{t}}$ *stopping time vector and* $\boldsymbol{\tau}(t+h) = \boldsymbol{\tau}(t) + \boldsymbol{\tau}(h) \circ \theta_{\boldsymbol{\tau}(t)}$ *for* $t, h \geqslant 0$.

Proof. To prove that $\boldsymbol{\tau}(t)$ is a stopping time vector we must show that

$$A = \bigcap_{n=1}^{N} \{\tau_n(t) \leqslant u_n\} \in \mathscr{F}_{\mathbf{u}}$$

for $\mathbf{u} = \langle u_n \rangle \in \mathbf{T}$ having real components. It is known (cf. [1]) that there is a sequence $\mathbf{t}_i$ in $\mathbf{T}$ and a real-valued measurable function F such that

$$I_A = F(\xi_{\mathbf{t}_1}, \xi_{\mathbf{t}_2}, \ldots),$$

where I_A is the indicator function of the set A. Suppose ω, $\tilde{\omega} \in \Omega$ and $\xi_{\mathbf{s}}(\omega) = \xi_{\mathbf{s}}(\tilde{\omega})$ for all $\mathbf{s} \leqslant \mathbf{u}$. Then

$$F(\xi_{\mathbf{t}_1}(\omega), \xi_{\mathbf{t}_2}(\omega), \ldots) = F(\xi_{\mathbf{t}_1}(\tilde{\omega}), \xi_{\mathbf{t}_2}(\tilde{\omega}), \ldots)$$

by Lemma 8. Now choose $\tilde{\omega}$ such that $\xi_{\mathbf{s}}(\tilde{\omega}) = \xi_{\mathbf{s}\wedge\mathbf{u}}(\omega)$ for all $\mathbf{s} \in \mathbf{T}$. Then

$$F(\xi_{\mathbf{t}_1}(\omega), \xi_{\mathbf{t}_2}(\omega), \ldots) = F(\xi_{\mathbf{t}_1\wedge\mathbf{u}}(\omega), \xi_{\mathbf{t}_2\wedge\mathbf{u}}(\omega), \ldots)$$

for all $\omega \in \Omega$. This shows that $A = \bigcap_{n=1}^{N} \{\tau_n(t) \leqslant u_n\} \in \mathscr{F}_{\mathbf{u}}$ for any $\mathbf{u} \in \mathbf{T}$. We now show that $\tau_n(t+h) = \tau_n(t) + \tau_n(h) \circ \theta_{\boldsymbol{\tau}(t)}$ for each n whenever $t, h \geqslant 0$. To do this, we will fix t and let h vary. Also fix $\omega \in \Omega$ and let $\tilde{\omega} = \theta_{\boldsymbol{\tau}(t)}\omega = \theta_{\boldsymbol{\tau}(t)(\omega)}(\omega)$. If we let $\sigma_n(h)(\omega) = \tau_n(t+h)(\omega)$, then the $\sigma_n(\cdot)(\omega)$ satisfy the initial value problem

$$\begin{aligned} \dot{\sigma}_n(h)(\omega) &= c_n(\xi^1_{\sigma_1(h)(\omega)}(\omega), \ldots, \xi^N_{\sigma_N(h)(\omega)}(\omega)), \qquad h \geqslant 0 \\ \sigma_n(0)(\omega) &= \tau_n(t)(\omega). \end{aligned} \tag{4.5}$$

Letting $\mu_n(h) = \tau_n(t)(\omega) + \tau_n(h)(\tilde{\omega})$,

$$\begin{aligned} \dot{\mu}_n(h) &= (d/dh)[\tau_n(h)(\tilde{\omega})] \\ &= c_n(\xi^1_{\tau_1(h)(\tilde{\omega})}(\tilde{\omega}), \ldots, \xi^N_{\tau_N(h)(\tilde{\omega})}(\tilde{\omega}) \\ &= c_n(\xi^1_{\tau_1(h)(\tilde{\omega})}(\theta_{\boldsymbol{\tau}(t)}\omega), \ldots, \xi^N_{\tau_N(h)(\tilde{\omega})}(\theta_{\boldsymbol{\tau}(t)}\omega)) \\ &= c_n(\xi^1_{\mu_1(h)(\omega)}(\omega), \ldots, \xi^N_{\mu_N(h)(\omega)}(\omega)). \end{aligned}$$

Since $\mu_n(0) = \tau_n(t)(\omega)$, the functions $\mu_n(h)$ also satisfy (4.5). In view of the uniqueness of the solution of (4.2), $\mu_n(h) = \sigma_n(h)$, $h \geqslant 0$, or what is the same, $\tau_n(t+h)(\omega) = \tau_n(t)(\omega) + \tau_n(h)(\theta_{\boldsymbol{\tau}(t)}\omega)$.

Under the conditions imposed upon the speed functions c_n, the solution $\tau(t) = \langle\tau_n(t)\rangle$ of (4.2) satisfies the conditions of the preceding section for using the $\tau_n(t)$ as simultaneous time substitutions. Letting $\eta_t = \xi_{\tau(t)}$, the process $\{\Omega, \mathscr{F}, \mathscr{F}_{\tau(t)+}, \eta_t, P_x\}$ is a strong Markov process with right-continuous paths, in fact, a step process. It need not be true, in general, that the η_t process inherits the Feller property of the ξ_t process. This point has been discussed by Lamperti [5] for single component processes. We will bypass this problem as did Volkonskii [7].

THEOREM 10. *If the process $\{\Omega, \mathscr{F}, \mathscr{F}_{\tau(t)+}, \eta_t, P_x\}$ is a Feller process, then the infinitesimal generator A of the semigroup of operators $\{P_t: t \geqslant 0\}$ defined for $f \in C(X)$ by $P_t f(x) = E_x[f(\eta_t)]$ is given by*

$$Af = \sum_{n=1}^{N} c_n A_n f, \qquad f \in C(X). \tag{4.6}$$

Proof. Since the A_n are bounded operators on $C(X)$, it suffices to prove the result for $f \in C(X)$ of the form $f = \prod_{n=1}^{N} f_n$, where each f_n is a continuous function of just the n-th component x_n of x, and A replaced by A_w, the weak infinitesimal generator of the η_t process (cf. [3, p. 143]). Consider any $x = \langle x_n\rangle \in X$. Let σ_x be the first time the η_t process leaves the point x, and let $\sigma_{x,j} = \sigma_x \wedge j^{-1}$. It is known that $P_x[\sigma_x > t] = e^{-at}$ for each $t \geqslant 0$ and some $0 \leqslant a \leqslant +\infty$, depending possibly upon x. The constant a cannot be $+\infty$ for this would imply that $\sigma_x = 0$ a.e. P_x in contradiction to the fact that the η_t process is a step process. On the other hand, if $a = 0$, then $\sigma_x = +\infty$ a.e. P_x, and this would mean that the trajectories of the η_t process, and therefore the ξ_t process, never leave the point x; for the η_t process this would imply that $Af(x) = 0$ while for the components of the ξ_t process this would imply that $A_n f(x) = 0$ (cf. [3, p. 137]), thereby establishing the validity of (4.6) at x. We can therefore assume that $0 < a < +\infty$ and, in particular, that $0 < E_x[\sigma_{x,j}] < \infty$. It is also known that

$$A_w f(x) = \lim_{j\to\infty} (E_x[f(\eta_{\sigma_{x,j}})] - f(x))/E_x[\sigma_{x,j}]$$

provided the right side is continuous at x (cf. [3, p. 143]). Since $\sigma_{x,j}$ is an $\mathscr{F}_{\tau(t)+}$ stopping time, $\tau(\sigma_{x,j})$ is an $\mathscr{F}_{t+}$ stopping time vector by Lemma 6. By definition, $\eta_{\sigma_{x,j}} = \xi_{\tau(\sigma_{x,j})}$ and

$$A_w f(x) = \lim_{j\to\infty} (E_x[f(\xi_{\tau(\sigma_{x,j})})] - f(x))/E_x[\sigma_{x,j}].$$

By Theorem 5 and the fact that $\tau_n(\sigma_{x,j}) \leqslant M\sigma_{x,j} \leqslant Mj^{-1}$,

$$\frac{E_x[f(\xi_{\tau(\sigma_{x,j})})] - f(x)}{E_x[\sigma_{x,j}]} = \frac{E_x\left[\prod_{n=1}^{N} f_n(\xi^n_{\tau_n(\sigma_{x,j})})\right] - f(x)}{E_x[\sigma_{x,j}]}$$

$$= o(1) + \sum_{n=1}^{N} \frac{E_x\left[\prod_{m=1,m\neq n}^{N} f_m(\xi^m_{\tau_m(\sigma_{x,j})}) \int_0^{\tau_n(\sigma_{x,j})} A_n f_n(\xi_t{}^n)\,dt\right]}{E_x[\tau_n(\sigma_{x,j})]}$$

$$\times \frac{E_x[\tau_n(\sigma_{x,j})]}{E_x[\sigma_{x,j}]}. \tag{4.7}$$

For fixed $\omega \in \Omega$, consider the integral

$$\int_0^{\tau_n(\sigma_{x,j})(\omega)} A_n f_n(\xi_t{}^n(\omega))\,dt.$$

Suppose $0 < t < \tau_n(\sigma_{x,j})(\omega)$. There is then an $s < \sigma_{x,j}(\omega)$ such that $\tau_n(s) = t$. Since $\eta_s{}^n(\omega) = \xi^n_{\tau_n(s)}(\omega) = \xi_t{}^n(\omega)$ and $\eta_s(\omega) = x$,

$$\int_0^{\tau_n(\sigma_{x,j})(\omega)} A_n f_n(\xi_t{}^n(\omega))\,dt = A_n f_n(x_n)\,\tau_n(\sigma_{x,j})(\omega).$$

We will now show that the first factor of the general term in the sum in (4.7) has limit $A_n f_n(x_n) \prod_{m=1,m\neq n}^{N} f_m(x_m)$ as $j \to \infty$. Now

$$\left| \frac{E_x\left[\left(\prod_{\substack{m=1\\m\neq n}}^{N} f_m(\xi^m_{\tau_m(\sigma_{x,j})})\right) A_n f_n(x_n)\,\tau_n(\sigma_{x,j})\right]}{E_x[\tau_n(\sigma_{x,j})]} - A_n f_n(x_n) \prod_{\substack{m=1\\m\neq n}}^{N} f_m(x_m) \right|$$

$$\leqslant K_1 \frac{E_x\left[\left| \prod_{\substack{m=1\\m\neq n}}^{N} f_m(\eta^m_{\sigma_{x,j}}) - \prod_{\substack{m=1\\m\neq n}}^{N} f_m(x_m) \right| \tau_n(\sigma_{x,j})\right]}{E_x[\tau_n(\sigma_{x,j})]}$$

$$\leqslant K_2 \frac{E_x[I_{\{\sigma_x \leqslant j^{-1}\}} \cdot \sigma_{x,j}]}{E_x[\sigma_{x,j}]},$$

where K_1 and K_2 are constants depending only upon the f_n, m, and M. Using the fact that $P_x[\sigma_x > t] = e^{-at}$, the last quotient is easily seen to be aj^{-1} and goes to zero as $j \to \infty$. Consider the second factor of the general term of the sum in (4.7):

$$\frac{E_x[\tau_n(\sigma_{x,j})]}{E_x[\sigma_{x,j}]} = \frac{E_x\left[\int_0^{\sigma_{x,j}} c_n(\xi^1_{\tau_1(t)}, \ldots, \xi^N_{\tau_N(t)})\, dt\right]}{E_x[\sigma_{x,j}]} = \frac{E_x\left[\int_0^{\sigma_{x,j}} c_n(\eta_t)\, dt\right]}{E_x[\sigma_{x,j}]}$$

For $0 \leqslant t < \sigma_{x,j}$, $x = \eta_t$ a.e. P_x and $c_n(\eta_t) = c_n(x)$. The last quotient is therefore equal to $c_n(x)$. This proves that

$$\begin{aligned} A_w f(x) &= \sum_{n=1}^{N} c_n(x)\, A_n f_n(x_n) \prod_{\substack{m=1\\ m\neq n}}^{N} f_m(x_m) \\ &= \sum_{n=1}^{N} c_n(x)\, A_n f(x) \end{aligned}$$

since the sum on the right is continuous.

5. Ergodic Properties

In addition to all the assumptions made heretofore, we will now assume that the state space X is finite and endowed with the discrete topology. One consequence of this assumption is that any Markov process discussed herein is automatically a Feller process, thereby ensuring the applicability of Theorem 10. Given bounded operators A_n on $C(E_\Delta)$ and speed functions c_n, we can obtain an η_t process with generator $A = \sum_{n=1}^{N} c_n A_n$ from the ξ_t process by means of the random time change $\tau(t)$. In this section, we will relate the asymptotic distribution $\lim_{t\to\infty} P_x[\eta_t = y]$ to the asymptotic distributions of the components of the ξ_t process and the speed functions c_n.

Since the spaces E_Δ and X are finite, the operators A_n and A are really matrix operators, and we will employ the usual notation of kernels by letting $A_n f_n(x_n) = \sum_{y_n} f_n(y_n)\, A_n(x_n, y_n)$ for each real-valued function f_n on E_Δ and $Af(x) = \sum_{y\in X} f(y)\, A(x, y)$ for each real-valued function f on X where $A_n(x_n, y_n)$ and $A(x, y)$ are defined as $A_n(x_n, y_n) = A_n I_{\{y_n\}}(x_n)$ and $A(x, y) = AI_{\{y\}}(x)$, respectively.

For the time being, consider just one of the ξ_t^n processes and the associated Markovian semigroup of kernels P_t^n. For each $x_n, y_n \in E_\Delta$, either $P_t^n(x_n, y_n) = 0$ for all t or $P_t^n(x_n, y_n) > 0$ for all $t > 0$; in the latter case, y_n is said to be accessible from x_n, and the relation of accessibility gives a decomposition of the state space E_Δ into a class F^n of transient states and ergodic classes $E_1{}^n,\ldots, E_{k_n}^n$. If $y_n \neq x_n$ and y_n is not accessible from x_n, then $P_t(x_n, y_n) = 0$ for all $t > 0$ and $A_n(x_n, y_n) = 0$; in other words, if $y_n \neq x_n$ and $A_n(x_n, y_n) \neq 0$, then y_n is accessible from x_n. For each $x_n \in E_\Delta$ and ergodic class $E_j{}^n$, we will put

$$\rho_{x_n}^n(E_j{}^n) = \lim_{t\to\infty} P_{x_n}^n[\xi_t{}^n \in E_j{}^n].$$

If $x_n \in E_j{}^n$, then $\rho_{x_n}^n(E_j{}^n) = 1$ since the $\xi_t{}^n$ process can never leave $E_j{}^n$ if it starts from a point of $E_j{}^n$ a.e. $P_{x_n}^n$. With each ergodic class $E_j{}^n$ there is associated a probability measure $\mu_j{}^n$ having (proper) support $E_j{}^n$ such that

$$\lim_{t\to\infty} P_{x_n}^n[\xi_t{}^n = y_n] = \begin{cases} 0 & \text{if } \ y_n \in F^n \\ \mu_j{}^n(y_n) & \text{if } \ x_n \in E_j{}^n \\ \rho_{x_n}^n(E_j{}^n)\, \mu_j{}^n(y_n) & \text{if } \ x_n \in F^n, y_n \in E_j{}^n. \end{cases}$$

We will assume that the measures $\mu_1{}^n,\ldots, \mu_{k_n}^n$ and the $\rho_{x_n}^n(E_j{}^n)$ are known; that is, the asymptotic distribution $\lim_{t\to\infty} P_{x_n}^n[\xi_t{}^n = y_n]$ of the $\xi_t{}^n$ process is completely known. We will identify the asymptotic distribution of the η_t process in terms of the measures $\mu_1{}^n ,\ldots, \mu_{k_n}^n$ and the $\rho_{x_n}^n(E_j{}^n)$, $n = 1,\ldots, N$.

LEMMA 11. *If $x \in X$ and $E_{i_n}^n$ is an ergodic class for the $\xi_t{}^n$ process, $n = 1,\ldots, N$, then*

$$P_x[\xi_{\mathbf{s}} \in E_{i_1}^1 \times \cdots \times E_{i_N}^N, \xi_{\mathbf{s}+\mathbf{t}} \notin E_{i_1}^1 \times \cdots \times E_{i_N}^N \textit{ for some } \mathbf{s}, \mathbf{t} \geqslant 0] = 0$$

Proof. Let $\mathbf{Q} = \{\mathbf{t} = \langle t_n\rangle \in \mathbf{T}: t_n \in Q\}$, the above probability is less than or equal to

$$\sum_{\mathbf{s},\mathbf{t}\in\mathbf{Q}} P_x[\xi_{\mathbf{s}} \in E_{i_1}^1 \times \cdots \times E_{i_N}^N, \xi_{\mathbf{s}+\mathbf{t}} \notin E_{i_1}^1 \times \cdots \times E_{i_N}^N]$$

$$= \sum_{\mathbf{s},\mathbf{t}\in\mathbf{Q}} \sum_{\substack{y\in E_{i_1}^1\times\cdots\times E_{i_N}^N \\ z\notin E_{i_1}^1\times\cdots\times E_{i_N}^N}} P_x[\xi_{\mathbf{s}} = y, \xi_{\mathbf{s}+\mathbf{t}} = z].$$

Since $P_x[\xi_{\mathbf{s}} = y,\ \xi_{\mathbf{t}+\mathbf{s}} = z] = \prod_{n=1}^N P_{x_n}^n[\xi_{s_n}^n = y_n, \xi_{s_n+t_n}^n = z_n]$ and at least one of the factors is zero whenever $y \in E_{i_1}^1 \times \cdots \times E_{i_N}^N$ and $z \notin E_{i_1}^1 \times \cdots \times E_{i_N}^N$, the above sums are also zero.

LEMMA 12. *For each* $x = \langle x_n \rangle \in X$,

$$\prod_{n=1}^{N} P_{x_n}[\xi_{mt}^n \in E_{i_n}^n] \leqslant P_x[\eta_t \in E_{i_1}^1 \times \cdots \times E_{i_N}^N] \leqslant \prod_{n=1}^{N} P_{x_n}[\xi_{Mt}^n \in E_{i_n}^n]$$

and

$$\lim_{t\to\infty} P_x[\eta_t \in E_{i_1}^1 \times \cdots \times E_{i_N}^N] = \prod_{n=1}^{N} \rho_{x_n}^n(E_{i_n}^n).$$

Proof. Letting

$$Z = \{\omega : \xi_{\mathbf{s}}(\omega) \in E_{i_1}^1 \times \cdots \times E_{i_N}^N,\ \xi_{\mathbf{s}+\mathbf{t}}(\omega) \notin E_{i_1}^1 \times \cdots \times E_{i_N}^N \text{ for some } \mathbf{s}, \mathbf{t} \geqslant 0\},$$
$$P_x[Z] = 0.$$

For $\omega \notin Z$, $\mathbf{s} \geqslant \mathbf{0}$, and $\mathbf{t} \geqslant \mathbf{s}$,

$$\xi_{\mathbf{s}}(\omega) \in E_{i_1}^1 \times \cdots \times E_{i_N}^N \Rightarrow \xi_{\mathbf{t}}(\omega) \in E_{i_1}^1 \times \cdots \times E_{i_N}^N.$$

Since $\langle mt \rangle \leqslant \boldsymbol{\tau}(t) \leqslant \langle Mt \rangle$, for $\omega \notin Z$,

$$\begin{aligned} \xi_{\langle mt \rangle}(\omega) \in E_{i_1}^1 \times \cdots \times E_{i_N}^N &\Rightarrow \xi_{\boldsymbol{\tau}(t)(\omega)}(\omega) \in E_{i_1}^1 \times \cdots \times E_{i_N}^N \\ &\Rightarrow \xi_{\langle Mt \rangle}(\omega) \in E_{i_1}^1 \times \cdots \times E_{i_N}^N. \end{aligned}$$

Since $P_x[Z] = 0$,

$$\begin{aligned} \prod_{n=1}^{N} P_{x_n}[\xi_{mt}^n \in E_{i_n}^n] &= P_x[\xi_{\langle mt \rangle} \in E_{i_1}^1 \times \cdots \times E_{i_N}^N] \\ &\leqslant P_x[\xi_{\boldsymbol{\tau}(t)} \in E_{i_1}^1 \times \cdots \times E_{i_N}^N] \\ &\leqslant P_x[\xi_{\langle Mt \rangle} \in E_{i_1}^1 \times \cdots \times E_{i_N}^N] \\ &= \prod_{n=1}^{N} P_{x_n}[\xi_{Mt}^n \in E_{i_n}^n] \end{aligned}$$

Since $P_x[\eta_t \in E_{i_1}^1 \times \cdots \times E_{i_N}^N] = P_x[\xi_{\boldsymbol{\tau}(t)} \in E_{i_1}^1 \times \cdots \times E_{i_N}^N]$, we get the result by letting $t \to \infty$.

LEMMA 13. *If* $x, y \in E_{i_1}^1 \times \cdots \times E_{i_N}^N$, *then* $P_x[\eta_t = y] > 0$ *for all* $t > 0$.

Proof. If $x = y$, then $P_x[\eta_t = y] > 0$ for all $t > 0$ since

$$\lim_{t \to 0^+} P_x[\eta_t = y] = 1.$$

Suppose $y \neq x$ and differs from $x = \langle x_n \rangle$ in just one component, say the p-th component. Regarding A_p as a kernel on $E_\Delta \times E_\Delta$, there is a constant $K > 0$ such that $B_p = I + K^{-1}A_p \geqslant 0$, where I is the identity kernel on $E_\Delta \times E_\Delta$. Note that $B_p^r \geqslant 0$ for all $r \geqslant 1$. Then

$$P_t^p = e^{tA_p} = e^{-Kt} e^{Kt(I+K^{-1}A_p)} = e^{-Kt} e^{KtB_p}$$

and

$$P_t^p = e^{-Kt} \sum_{r=0}^{\infty} \frac{(Kt)^r}{r!} B_p^r.$$

Since $x_p, y_p \in E_{i_p}^p$,

$$P_{x_p}^p[\xi_t^p = y_p] = P_t^p(x_p, y_p) = e^{-Kt} \sum_{r=0}^{\infty} \frac{(Kt)^r}{r!} B_p^r(x_p, y_p) > 0$$

for all $t > 0$ and there is an integer $q \geqslant 1$ such that $B_p^q(x_p, y_p) > 0$. Since

$$B_p^q(x_p, y_p) = \sum_{\alpha_1, \ldots, \alpha_{q-1} \in E_\Delta} B_p(x_p, \alpha_1) B_p(\alpha_1, \alpha_2) \cdots B_p(\alpha_{q-1}, y_p) > 0,$$

there are $\alpha_1, \alpha_2, \ldots, \alpha_{q-1} \in E_\Delta$ such that

$$B_p(x_p, \alpha_1) B_p(\alpha_1, \alpha_2) \cdots B_p(\alpha_{q-1}, y_p) > 0.$$

If any two successive elements in the chain $x_p, \alpha_1, \alpha_2, \ldots, y_p$ are equal, then the corresponding factor in the product on the left can be deleted. We might as well assume that no two successive elements are equal. Since $B_p(\alpha, \beta) = K^{-1}A_p(\alpha, \beta)$ whenever $\alpha, \beta \in E_\Delta$, $\alpha \neq \beta$,

$$K^{-q}[A_p(x_p, \alpha_1) A_p(\alpha_1, \alpha_2) \cdots A_p(\alpha_{q-1}, y_p)] > 0.$$

Since c_p is strictly positive on X,

$$c_p(x) A_p(x_p, \alpha_1) c_p(s_p(\alpha_1)x) A_p(\alpha_1, \alpha_2) \cdots c_p(s_p(\alpha_{q-1})x) A_p(\alpha_{q-1}, y_p) > 0, \quad (5.1)$$

where $s_p(\alpha)x = s_p(\alpha)(x_1, \ldots, x_N) = (x_1, \ldots, x_{p-1}, \alpha, x_{p+1}, \ldots, x_N)$. Now consider the kernel A on $X \times X$ and the semigroup $P_t = e^{tA}$. Choose $C > 0$ such that $I + C^{-1}A \geqslant 0$. Then

$$P_t = e^{tA} = e^{-Ct}e^{Ct(I+C^{-1}A)} = e^{-Ct}\sum_{r=0}^{\infty}\frac{(Ct)^r}{r!}(I + C^{-1}A)^r$$

and

$$P_x[\eta_t = y] = e^{-Ct}\sum_{r=0}^{\infty}\frac{(Ct)^r}{r!}(I + C^{-1}A)^r(x, y)$$

with each term of the series nonnegative. By (5.1),

$$\begin{aligned}(I &+ C^{-1}A)^q(x, y)\\ &\geqslant C^{-q}A(x, s_p(\alpha_1)x)\,A(s_p(\alpha_1)x, s_p(\alpha_2)x)\cdots A(s_p(\alpha_{q-1})x, y)\\ &= C^{-q}c_p(x)\,A_p(x_p, \alpha_1)\,c_p(s_p(\alpha_1)x)\,A_p(\alpha_1, \alpha_2)\cdots c_p(s_p(\alpha_{q-1})x)\,A_p(\alpha_{q-1}, y_p) > 0.\end{aligned}$$

It follows that $P_x[\eta_t = y] > 0$ for all $t > 0$ whenever y differs from x in exactly one component. Clearly, $P_x[\eta_t = y] > 0$ for all $t > 0$ whenever $x, y \in E^1_{i_1} \times \cdots \times E^N_{i_N}$.

LEMMA 14. *$F = X \sim \bigcup_{i_1,\ldots,i_N} E^1_{i_1} \times \cdots \times E^N_{i_N}$ is the class of transient states for the η_t process.*

Proof. Consider any $x \in F$. By Lemma 12, $\lim_{t\to\infty} P_x[\eta_t \in \sim F] = 1$. This means that there is a $y \in E^1_{i_1} \times \cdots \times E^N_{i_N}$ for some $i_1, \ldots, i_N$ such that $\lim_{t\to\infty} P_x[\eta_t = y] > 0$; in particular, $P_t(x, y) = P_x[\eta_t = y] > 0$ for all $t > 0$. On the other hand, $P_t(y, x) = P_y[\eta_t = x] = 0$ for all $t > 0$ because the η_t process must remain in $E^1_{i_1} \times \cdots \times E^N_{i_N}$ with P_y measure 1 according to Lemma 12. This shows that any $x \in F$ is a transient state for the η_t process. Now assume that $x \in \sim F$ is a transient state for the η_t process. Then $x \in E^1_{i_1} \times \cdots \times E^N_{i_N}$ for some choice of $i_1, \ldots, i_N$ and there is a y such that $P_x[\eta_t = y] > 0$ for all $t > 0$ but $P_y[\eta_t = x] = 0$ for all $t > 0$. Since $x \in E^1_{i_1} \times \cdots \times E^N_{i_N}$, the η_t process must remain in $E^1_{i_1} \times \cdots \times E^N_{i_N}$ with P_x measure 1 and therefore $y \in E^1_{i_1} \times \cdots \times E^N_{i_N}$. But by Lemma 13, $P_y[\eta_t = x] > 0$ for all $t > 0$, a contradiction. Therefore, no $x \in \sim F$ can be a transient state for the η_t process.

The two preceding results identify the ergodic classes of the η_t process in terms of the ergodic classes of the ξ^n_t processes. Recalling that $\pi_n x = (x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N)$ whenever $x = \langle x_n \rangle$, we can now state the principal result.

THEOREM 15. *If Ψ_n and Φ are positive functions on $\prod_{N=1}^{N-1} E_\Delta$ and X, respectively, and $c_n(x) = \Psi_n(\pi_n x)/\Phi(x)$, $n = 1,\dots, N$, then*

$$\lim_{t\to\infty} P_x[\eta_t = y] = \begin{cases} 0 & \text{if } y \in F \\ \gamma_{i_1,\dots,i_N}\Phi(y) \prod_{n=1}^{N} \mu^n_{i_n}(y_n) & \text{if } x \in E^1_{i_1} \times \cdots \times E^N_{i_N} \\ \gamma_{i_1,\dots,i_N} \prod_{n=1}^{N} \rho^n_{x_n}(E^n_{i_n})\, \Phi(y) \prod_{n=1}^{N} \mu^n_{i_n}(y_n) & \text{if } x \in F, \\ & y \in E^1_{i_1} \times \cdots \times E^N_{i_N}, \end{cases}$$

where $\gamma_{i_1,\dots,i_N}$ is a normalizing constant, $1 \leqslant i_j \leqslant k_j$, $j = 1,\dots, N$.

Proof. Consider a fixed ergodic class $E^1_{i_1} \times \cdots \times E^N_{i_N}$ of the η_t process. Since the measure

$$\mu(y) = \Phi(y) \prod_{n=1}^{N} \mu^n_{i_n}(y_n)$$

has (proper) support $E^1_{i_1} \times \cdots \times E^N_{i_N}$, we need only show that μ is an invariant measure for the η_t process; that is, that

$$\mu A(x) = \sum_{y\in X} \mu(y)\, A(y, x) = 0$$

for each $x \in X$. To do this, let f be any function on X. By (4.6)

$$Af(y) = \sum_{n=1}^{N} c_n(y)\, A_n f(y)$$

and

$$\begin{aligned} \sum_{y\in X} \mu(y)\, Af(y) &= \sum_{y\in X} \sum_{n=1}^{N} \mu(y)\, c_n(y)\, A_n f(y) \\ &= \sum_{n=1}^{N} \sum_{y\in X} \mu(y)\, c_n(y)\, A_n f(y) \\ &= \sum_{n=1}^{N} \sum_{y\in X} \left(\Phi(y) \prod_{m=1}^{N} \mu^m_{i_m}(y_m)\right) (\Psi_n(\pi_n y)/\Phi(y))\, A_n f(y) \\ &= \sum_{n=1}^{N} \sum_{y\in X} \Psi_n(\pi_n y) \prod_{m=1}^{N} \mu^m_{i_m}(y_m)\, A_n f(y). \end{aligned}$$

Since $\Psi_n(\pi_n y)$ does not depend upon y_n,

$$\sum_{y\in X} \mu(y)\, Af(y) = \sum_{n=1}^{N} \sum_{y_1,\ldots,y_{n-1},y_{n+1},\ldots,y_N} \Psi_n(\pi_n y) \prod_{\substack{m=1\\ m\neq n}}^{N} \mu_{i_m}^m(y_m) \sum_{y_n} \mu_{i_n}^n(y_n)\, A_n f(y).$$

Now

$$\begin{aligned}\sum_{y_n} \mu_{i_n}^n(y_n)\, A_n f(y) &= \sum_{y_n} \mu_{i_n}^n(y_n) \sum_{z_n} f(y_1,\ldots,y_{n-1}, z_n, y_{n+1},\ldots,y_N)\, A_n(y_n, z_n)\\ &= \sum_{z_n} f(y_1,\ldots,y_{n-1}, z_n, y_{n+1},\ldots,y_N) \sum_{y_n} \mu_{i_n}^n(y_n)\, A_n(y_n, z_n).\end{aligned}$$

Since $\mu_{i_n}^n$ is an invariant measure for the ξ_t^n process,

$$\mu_{i_n}^n A_n(z_n) = \sum_{y_n} \mu_{i_n}^n(y_n)\, A_n(y_n, z_n) = 0.$$

Therefore,

$$\sum_{y\in X} \mu(y)\, Af(y) = 0$$

for any real-valued function f on X. Taking f to be the indicator function of $\{x\}$,

$$\sum_{y\in X} \mu(y)\, A(y, x) = 0;$$

that is, μ is an invariant measure for the η_t process. Since any invariant measure of the η_t process is a convex linear combination of the ergodic measures associated with the ergodic classes of the η_t process and μ has (proper) support $E_{i_1}^1 \times \cdots \times E_{i_N}^N$, μ is proportional to the ergodic measure associated with $E_{i_1}^1 \times \cdots \times E_{i_N}^N$; choosing $\gamma_{i_1,\ldots,i_N}$ so that $\gamma_{i_1,\ldots,i_N}\mu$ is a probability measure, we must have

$$\lim_{t\to\infty} P_x[\eta_t = y] = \gamma_{i_1,\ldots,i_N}\mu(y)$$

whenever $x, y \in E_{i_1}^1 \times \cdots \times E_{i_N}^N$. The remainder of the theorem follows from Lemma 12.

References

1. R. M. Blumenthal and R. K. Getoor, "Markov Processes and Potential Theory," Academic, New York, 1968.
2. P. Courrège and P. Priouret, Temps d'arrêt d'une fonction aléatoire: relations

d'équivalence associées et propriétés de décomposition, *Publ. Inst. Stat. Univ. Paris* **14** (1965), 245–274.
3. E. B. DYNKIN, "Markov Processes," Vol. I, Springer–Verlag, Berlin, 1965.
4. L. L. HELMS, Stochastic formation of hierarchies, *J. Appl. Prob.*, to appear.
5. J. LAMPERTI, "On Random Time Substitutions and the Feller Property," Proceedings of a Symposium Conducted by the Mathematics Research Center and the United States Army at the University of Wisconsin, Madison (J. Chover, Ed.), Wiley, New York, 1967.
6. F. SPITZER, Interaction of Markov Processes, *Adv. Math.* **5** (1970), 246–290.
7. V. A. VOLKONSKII, Random substitution of time in strong Markov processes, *Theory Prob. Appl.* **3** (1958), 310–325.